Higher Order Homogeneous ODEs

Reduction of order

Given an equation in standard form:

$y'' + P(x)y' + Q(x)y = 0$

And a first solution, a second solution can be found:

$$y_2 = y_1(x) \int \frac{e^{\int P(x)dx}}{y_1^2} dx$$

Homogeneous Linear Equations with Constant Coefficients

Given an equation:

$y'' + P(x)y' + Q(x)y = 0$

Find the auxiliary equation and solve for m:

$m^2 + Pm + Q = 0$

If distinct roots:

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

If repeated real roots:

$y = c_1 e^{mx} + c_2 x e^{mx}$

If complex conjugates, $\alpha \pm \beta i$:

$y = c_1 e^{\alpha x} cos(\beta x) + c_2 e^{\alpha x} sin(\beta x)$

Homogeneous Linear Equations With Special Cases (Superposition approach)

$y'' + P(x)y' + Q(x)y = f(x)$

Where

$f(x)$ is a:

- Polynomial
- e^{mx}
- sin/cos
- Any linear combination of the above

Steps:

Solve the associated homogeneous equation to find y_c

Find y_p associated with $f(x)$

Given $f(x) =$	c	$cx + c$	cx	x^c	$sin(cx)$ or $cos(cx)$	e^{cx}
$y_p =$	A	$Ax + B$	$Ax + B$	$Ax^2 + Bx + C$	$Asin(cx) + Bcos(cx)$	Ae^{cx}

Check y_p against the homogeneous equation and modify if necessary

Differentiate y_p and plug into the DE to solve A, B, C…

Finally:

$y = y_p + y_c$

Variation of Parameters

Can be used for any homogeneous linear equation

Steps

Put in standard form:

$$y'' + P(x)y' + Q(x)y = f(x)$$

Solve the associated equation:

$$m^2 + Pm + Q = 0$$

To find:

$$y_c = c_1y_1 + c_2y_2$$

Compute cross products:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \qquad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

Compute:

$$U_1 = \int \frac{W_1}{W} dx \qquad U_2 = \int \frac{W_2}{W} dx$$

Plug in:

$$y_p = U_1y_1 + U_2y_2$$

Finally:

$$y = y_c + y_p$$